

Density Matrix From Photon Number Tomography

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Abstract

We provide a simple analytic relation which connects the density operator of the radiation field with the number probabilities. The problem of experimentally "sampling" a general matrix elements is studied, and the deleterious effects of nonunit quantum efficiency in the detection process are analyzed showing how they can be reduced by using the squeezing technique. The obtained result is particularly useful for intracavity field reconstruction states.

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After the seminal paper of Vogel and Risken [1] and the experimental realization of their result by Smithey *et al.* [2] it became clear that the homodyne measurement of an electromagnetic field permits the reconstruction of the Wigner function of a quantum state, by just varying the phase of the local oscillator, since then named optical homodyne tomography. In Ref. [1] it was shown, indeed, that the rotated quadrature distribution may be expressed in terms of any s-parametrized Wigner function [3].

These papers give rise to a plethora of other papers [4, 5] where the limitations due to the detectors quantum efficiency η were also taken into account, and a fundamental limit was established showing the impossibility of the tomographic reconstruction for detectors quantum efficiency less than 0.5 [6]. Very recently it was shown [7] that, in principle, there is no such limit; however, in order to get the desired results for η arbitrarily small, one needs to employ a rather complicate mathematical procedure for loss error compensation, which in practice make still persistent a lower limit on η values because of numerical problems. Anyway, all the above mentioned papers rely on homodyne measurements, or any other phase dependent measurement process. While this work was in progress we became aware that Wallentowitz and Vogel [8] proposed the s-parametrized Wigner function reconstruction similar, in its essence, to the present one by using direct photon counting and contemporarely an analogous scheme was adopted by Banaszek and Wódkiewicz [9]; however, in the following we shall discuss the relevant differences with our work.

We shall show in this letter that the state retrieval by photon counting can be used not only when the output beam is mixed with a reference field at a beamsplitter, as in Refs. [8, 9], but also in a physical situation similar to the one proposed by Brune *et al.* [10] for the generation and measurement of a Schrödinger cat state allowing thus the possibility of reconstructing its density matrix and, more generally, the possibility

to reconstruct cavity QED field state when there is not a direct access to the field.

We shall show that the sampling of density matrix elements can be achieved by photon number measurements, for detectors quantum efficiency greater than 0.5, in any base but coordinate representation. This $\eta \geq 0.5$ limitation, however, can be beaten if we use a physical technique instead of applying the mathematical procedure of Ref. [7].

We shall call the procedure *Photon Number Tomography* because it permits the reconstruction of the density matrix elements, in a suitable representation, by just detecting the number of photons at the given reference field and then scanning both its phase and its amplitude; differently from the usual homodyne tomography where a marginal distribution is recorded by homodyne measurements and then scanning only the phase.

As claimed in Ref. [3], it is possible to write

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi} W(\alpha, s) \hat{T}(\alpha, -s), \quad (1)$$

where the s -ordered weight function $W(\alpha, s)$ may be identified with the quasiprobability distributions $Q(\alpha)$, $W(\alpha)$ and $P(\alpha)$ when the ordering parameter s assumes the values $-1, 0, 1$ respectively; while the operator \hat{T} represents the complex Fourier transform of the s -ordered displacement operator $\hat{D}(\xi, s) = \hat{D}(\xi) e^{s|\xi|^2/2}$, which can also be written as [11]

$$\hat{T}(\alpha, s) = \frac{2}{1-s} \hat{D}(\alpha) \left(\frac{s+1}{s-1} \right)^{\hat{a}^\dagger \hat{a}} \hat{D}^{-1}(\alpha). \quad (2)$$

On the other hand the weight function $W(\alpha, s)$ is the expectation value of the operator $\hat{T}(\alpha, s)$ [3], i.e. $W(\alpha, s) = \text{Tr}\{\hat{\rho} \hat{T}(\alpha, s)\}$, then we obtain

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi} \sum_{n=0}^{\infty} \frac{2}{1-s} \left(\frac{s+1}{s-1} \right)^n \langle n | \hat{D}(\alpha) \hat{\rho} \hat{D}^{-1}(\alpha) | n \rangle \hat{T}(-\alpha, -s). \quad (3)$$

Thus, the weight function $W(-\alpha, s)$ is related to the ability of measuring the quantity $\langle n | \hat{D}(\alpha) \hat{\rho} \hat{D}^{-1}(\alpha) | n \rangle$ by scanning the whole phase space [12], i.e. by just varying

α . Now, we may consider one mode of the radiation, whose state $\hat{\rho}$ one wants to reconstruct, contained inside a cavity and, immediately before the photon number measurement, a coherent reference field is "added" [10, 13], so that we may recognize

$$\mathcal{P}(n, \alpha) = \text{Tr}\{\hat{D}(\alpha)\hat{\rho}\hat{D}^{-1}(\alpha)|n\rangle\langle n|\} = \langle n|\hat{D}(\alpha)\hat{\rho}\hat{D}^{-1}(\alpha)|n\rangle \quad (4)$$

as the probability to detect n photons after the injection of the reference field α . The addition process we are considering, following Ref. [10], "is quite different from the combination of fields produced by a beam splitter, which mixes together distinct modes coupled to its two ports and introduces vacuum noise even in the absence of any classical input field". We are indeed describing a much simpler field amplitude superposition mechanism, discussed in the Glauber's pioneering work [13]. The photon number distribution (4) results as the projection of the field state $\hat{\rho}$ over a displaced number state [14].

The photon counting could be made either by means of atoms [15] as in the case of microwave cavity field or by direct detection of the outgoing optical total field. Furthermore, setting

$$\hat{K}_s(n, \alpha) = \frac{2}{1-s} \left(\frac{s+1}{s-1}\right)^n \hat{T}(-\alpha, -s), \quad (5)$$

we have, from Eq. (3)

$$\hat{\rho} = \sum_{n=0}^{\infty} \int \frac{d^2\alpha}{\pi} \mathcal{P}(n, \alpha) \hat{K}_s(n, \alpha). \quad (6)$$

Thus, analogously to Ref. [6], we may assert that a density matrix element can be experimentally sampled if there exist at least one value of the parameter s inside the range $[-1, 1]$ for which the corresponding matrix element of the kernel operator \hat{K}_s is bounded. From Eqs. (5) and (2), one immediately recognizes that this is possible for $s \in (-1, 0]$, in the number, coherent and squeezed representations and not in the coordinate basis. For $s = -1$, in Eq. (5) $n = 0$ only survives, however as was

shown in Ref. [3], the operator \hat{T} becomes singular and can be used to construct an arbitrary density matrix when is only weighted with the well behaved function Q . It means that $\mathcal{P}(0, \alpha) \equiv Q(\alpha)$ in Eq. (6).

Let us now consider the more realistic case of nonunit quantum efficiency η . Accordingly to Ref. [16] we have

$$\mathcal{P}(n, \alpha) = \eta^{-n} \sum_{k=0}^{\infty} \mathcal{P}_{\eta}(n+k, \alpha) \binom{n+k}{n} \left(\frac{1-\eta}{\eta} \right)^k (-1)^k. \quad (7)$$

where \mathcal{P}_{η} represents the same of (4) but in presence of efficiency η , i.e.

$$\mathcal{P}_{\eta}(n, \alpha) = \text{Tr} \left\{ \hat{D}(\alpha) \hat{\rho} D^{-1}(\alpha) : e^{-\eta \hat{a}^{\dagger} \hat{a}} \frac{(\eta \hat{a}^{\dagger} \hat{a})^n}{n!} : \right\}. \quad (8)$$

If we substitute Eq. (7) into Eq. (6), we obtain

$$\hat{\rho} = \sum_{m=0}^{\infty} \int \frac{d^2\alpha}{\pi} \mathcal{P}_{\eta}(m, \alpha) \left[\sum_{l=0}^m \eta^{-(m-l)} \left(\frac{\eta-1}{\eta} \right)^l \binom{m}{m-l} \hat{K}_s(m-l, \alpha) \right], \quad (9)$$

where the quantity inside the square brackets can be considered as a modified kernel

$$\hat{K}_{s,\eta}(m, \alpha) = \frac{2}{1-s} \left(\frac{\eta s - \eta + 2}{\eta s - \eta} \right)^m \hat{T}(-\alpha, -s). \quad (10)$$

Again, this kernel results bounded only in the number, coherent and squeezed representations, but in that case the s values for which that occurs are determined by the quantum efficiency; in particular we have from Eq. (10) $s \in (-1, -(1-\eta)/\eta]$, so that η should be greater than 0.5, similar to Ref. [6].

It is also interesting to note as was stressed in Ref. [8], that if one chooses $s = 1 - 2/\eta$ in Eq. (10), only the term with $m = 0$ survives in Eq. (9), and this means that the desired density matrix elements are simply related to the zero-count probability which gives only the Q -function. Unfortunately, only quantum efficiency $\eta = 1$ (i.e. $s = -1$) allows to get the Q -function in this simpler manner, otherwise, for $\eta < 1$, s takes forbidden values (i.e. becomes smaller than -1).

Let us now show how the above limit on detectors efficiency can be beaten. In case one squeezes the radiation inside the cavity immediately before the injection of the reference field, the superposition mechanism [13] can be read as

$$\hat{D}(\alpha)\hat{S}(\zeta)\hat{\rho}\hat{S}^{-1}(\zeta)\hat{D}^{-1}(\alpha) = \hat{D}(\alpha)\hat{\tilde{\rho}}\hat{D}^{-1}(\alpha), \quad (11)$$

where $\hat{S}(\zeta)$ is the well known squeeze operator [17] with the squeezing parameter $\zeta = |\zeta|e^{i\varphi}$. Of course one has to be sure that the squeezing process is like a kick during which the natural evolution of the system is negligible; only under this assumption Eq. (11) holds. It could be physically realized, for example, by using a δ -kicking of frequency of the cavity mode [18].

Since the weight function $W(\alpha, s)$ can be written as the complex Fourier transform of the characteristic function, let us examine the relation between the operator of Eq. (11) and $\hat{D}(\alpha)\hat{\tilde{\rho}}\hat{D}^{-1}(\alpha)$ in terms of characteristic functions. By definition we have

$$\tilde{\chi}(\xi, s) = \text{Tr}\{\hat{\tilde{\rho}}\hat{D}(\xi, s)\} = \text{Tr}\{\hat{\rho}\hat{S}^{-1}(\zeta)e^{\xi\hat{a}^\dagger}e^{-\xi^*\hat{a}}\hat{S}(\zeta)\}e^{-|\xi|^2/2}e^{s|\xi|^2/2}. \quad (12)$$

Remembering how the squeeze operator acts on the annihilation (creation) operator [17], we obtain

$$\tilde{\chi}(\xi, s) = \text{Tr}\{\hat{\rho}\hat{D}(\xi\mu^* + \xi^*\nu)\}e^{s|\xi|^2/2} = \chi(\xi\mu^* + \xi^*\nu)e^{s|\xi|^2/2}, \quad (13)$$

with $\mu = \cosh|\zeta|$ and $\nu = e^{i\varphi}\sinh|\zeta|$. Writing now the variable ξ in polar coordinate $\xi = |\xi|e^{i\phi}$ and locking the squeezing parameter phase with that of the reference field, i.e. $\varphi = 2\phi$, we get $\xi\mu^* + \xi^*\nu = \xi(\cosh|\zeta| + \sinh|\zeta|) = \xi e^{|\zeta|} = \xi\Delta$, such that $\chi(\xi, s) = \tilde{\chi}(\xi/\Delta, s\Delta^2)$ and $W(\alpha, s) = \Delta^2\tilde{W}(\Delta\alpha, s\Delta^2)$. Hence, from Eqs. (1), (3), (7) and the last expression, we get

$$\hat{\rho} = \sum_{n=0}^{\infty} \int \frac{d^2\alpha}{\pi} \mathcal{P}_\eta(n, \alpha) \hat{\mathcal{K}}_{s,\eta}(n, \alpha/\Delta), \quad (14)$$

where

$$\hat{\mathcal{K}}_{s,\eta}(n, \frac{\alpha}{\Delta}) = \frac{2}{1 - s\Delta^2} \left(\frac{\eta s\Delta^2 - \eta + 2}{\eta s\Delta^2 - \eta} \right)^n \hat{T}(-\frac{\alpha}{\Delta}, -s), \quad (15)$$

from which one immediately recognizes that the ordering parameter s in the power term, is now scaled by the factor Δ^2 with respect to the one in Eq. (10), thus it ranges in the interval $s \in (-1, -(1 - \eta)/\eta\Delta^2]$, allowing values of quantum efficiency lower than 0.5. It is then possible to define an effective quantum efficiency $\tilde{\eta} = \Delta^2/[\Delta^2 + (1 - \eta)/\eta]$ which could be close to unit for sufficiently large squeezing independently of the real quantum efficiency.

For an easier implementation of this scheme, we could choose in Eq. (15) $s = \Delta^{-2}(1 - 2/\eta)$, in order to eliminate the sum over n in Eq. (14), directly relating the density matrix elements in any allowed basis to the zero-count probability; but in this case, due to the presence of the factor Δ^{-2} , the allowed values of s are not $s = -1$ as in Ref. [8], which only permits the reconstruction of the more smoothed Q -function, rather the value of s can approach zero from below depending on the value of the squeezing parameter, independently of η .

Thus, by squeezing preventively the field in a cavity allows one to detect the state of radiation inside it even for low efficiency detection. The use of squeezing technique to reduce the effect of nonunit quantum efficiency was already proposed in Ref. [5], but we use it in a different way. In Ref. [5], indeed, antisqueezing preamplification of the signal at the beam splitter was proposed rather than the antisqueezed signal inside the cavity. We would stress the fact that our Photon Number Tomography scheme becomes especially useful in the intracavity optical tomography where other similar schemes [8, 9] are not applicable; in fact it can be adopted in a situation in which the photon number is measured indirectly using a "sequence" of atoms passing through the cavity [15], with the quantum efficiency, determined only by the duration of the measurement process (i.e. the length of the "sequence"), that could be very high [10]. In that case, the scheme has also the advantage of being QND. Finally our scheme results suitable for cavity QED characterization, like Ref. [19], allowing the

reconstruction of nonclassical states as well, which are extremely sensitive to losses and then their detection seem prohibitive by means of an outgoing field as in Refs. [8, 9].

At last, our approach could be considered an extension of the known mathematical principle of tomography. In fact given a density operator $\hat{\rho}$ and a group element \mathcal{G} , one can create different types of tomography if, by knowing the matrix elements $\langle x|\mathcal{G}\hat{\rho}\mathcal{G}^{-1}|x\rangle$ from measurements, is able to invert the formula expressing the density operator in terms of the above distribution (the x may denote either continuous or discrete eigenvalues). For the inversion procedure one can use the properties of summation or integration over group parameters \mathcal{G} . The known tomographies are just given by this construction; for Radon transform or homodyne tomography, \mathcal{G} is the rotation group, for symplectic tomography [20] it is the symplectic group, but in principle one can use other groups. The only problem is mathematical one to make the inversion and/or physical one to realize the transformation \mathcal{G} in laboratory.

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